ABSTRACT A theoretical model is used to calculate the trajectory of a liquid jet in subsonic gaseous cross flows (JICF). In this model the jet cross section deforms due to the gas flow and changes from circle to ellipses. The aspect ratio of the ellipse grows larger as the time passes from the injection point. The jet trajectory is calculated by performing a force balance analysis of the liquid column’s cross section. This force analysis counts for aerodynamic, viscous and surface tension forces. The jet’s cross section is also reduced in area due to mass stripping from the liquid column. This mass reduction has a considerable effect on the jet trajectory and deformation at higher Weber numbers. The Weber number also plays a governing role in determination of the breakup regime which affects the rate and amount of mass stripping. Gas pressure is also believed to have considerable effect on the mass reduction and thus, the jet penetration. The drag coefficient of the cylinder changes considerably with Reynolds number and the jet deformation so, the drag coefficients of the elliptical cylinders of different cross sections are calculated numerically for Reynolds numbers between 200 and 8000 and for different ellipse aspect ratios.

Keywords: Crossflow, Mass stripping, Drag Force, Jet Breakup

1. INTRODUCTION

Jet in cross flow atomization problem involves very complex flow physics and has applications in fuel injection systems such as gas turbines, afterburners, augmenters and ramjet/scramjet combustors. Strong vortical structures, stripping of small droplets from the jet surface, and formation of differently sized ligaments and droplets make it economically not practical to perform a complete numerical simulation of this type of atomization yet, especially for industrial applications. These issues signal the demand for some simpler, yet reliable, models that can be used for industrial design purposes and can take into account important parameters such as flow conditions and physical properties of the liquid and gas phases.

In this study, we will study the effect of different parameters on this type of atomization using a theoretical approach. Details of the theoretical model are provided in [1].

2. MODEL

The model presented here counts for the deformation of the jet’s cross section, the variation in the drag coefficient and the mass reduction from the liquid column. The jet trajectory is then calculated considering the effect of the mentioned parameters.

2.1 Jet Deformation and Trajectory

We assume a cylindrical element of infinitesimal thickness $h$ from the liquid column. The element is deformed and moved along the jet trajectory due to the interplay between aerodynamic, surface tension and viscous forces. Here, we make the assumption that the jet cross section changes from a circle to an ellipse. At any point, the element is perpendicular to the path that its center of mass travels. The spatial location of this element as it moves along the jet trajectory is illustrated in Fig. 1.

The aspect ratio $e = a/b$ of the ellipse, which is defined as the ratio between ellipse’s semimajor $a$ and semiminor $b$ axes, increases with time till the breakup location. The assumptions made help approximating the liquid jet’s behavior. In reality, the jet cross section changes slightly into a kidney shape [2]. Clark [3] proposed that the deformation of a 2D liquid drop is dependant on the viscous, interfacial and inertial forces. His model was based on the analogy between an oscillating two-dimensional drop and a forced mass-spring system, as shown in Fig. 2. Inamura [4] performed a force balance analysis to predict the jet trajectory using the Clark formulations. His model predicted the jet trajectory up to locations far behind the jet breakup. Applying the Clark’s linearized formulation (for small oscillations of 2D liquid drops) to large deformations of the liquid jet, assuming the constant value of unity for the liquid column’s drag coefficient, neglecting the effect of mass reduction and ignoring the effect of liquid column’s curvature on the drag force were the most important issues that were not addressed in Inamura’s model and will be considered in the following article.

Using the same analogy as Clark, we will perform a force balance in the $x_2$ direction (Fig. 2) in the cross sectional plane to obtain the equation for the oscillation of the element.

Writing the force balance in the $x_2$ direction, we obtain the following relation:

$$F_p + F_r + F_y = m_{element} \ddot{y}$$  

(1)

Formulating the element deformation in terms of the motion of the center of mass of the half-element, the viscous force can be written in the following format [1]:

$$F_y = \frac{1}{2} \rho \pi a b \dot{y} v$$

where

- $\rho$ is the density of the liquid
- $a$ and $b$ are the semimajor and semiminor axes of the ellipse
- $\dot{y}$ is the velocity of the center of mass
- $v$ is the tangential velocity of the edge of the ellipse
\[ F_v = -2\pi \mu \rho_{eq} h \left( \frac{dv}{dt} \right) \]  

where \( r_{eq} \) is the radius of a circle with equal area to that of the instantaneous elliptical cross section. Surface tension force acting through the center of mass of the half element would be [1]:

\[ F_s = -\sigma \times h \times \frac{3\pi}{8} \left[ 4(1-r^2 a^{-2}) + \frac{c}{d} \right] \]  

where \( c \) and \( d \) are:

\[
\begin{align*}
    c &= 2r^2 (4 - \pi)(a^{m-1} - r^2 a^{-m-1}) \\
    d &= 2(a^m + r^2 a^{-m}) \frac{1}{(a^{m+1})} 
\end{align*}
\]

And finally the external force can be written in the form of [1]:

\[ F_p = \frac{1}{2} \rho \rho \frac{h h}{u u} \cos(\theta) \]  

where \( \theta \) is the angle between the vertical direction and the normal vector to the element. Velocity projection is vital since assuming a constant free stream velocity leads to overestimating the total force exerted on the element.

Substituting equations (2), (3) and (4) into (1), assuming \( m_{element} = 0.5 \rho_l \pi a b h \), and dividing all the terms by the element thickness \( h \) yield to the nonlinear equation of oscillation in the form of:

\[ c_1 \left( \frac{d^2 y}{dt^2} \right) + c_2 \left( \frac{dy}{dt} \right) + c_1 = c_4 \]  

where:

\[ c_1 = \frac{1}{2} \rho \rho \pi a b \\
\]

\[ c_2 = \frac{2\pi \mu \rho a b}{y^2} \]

\[ c_4 = \sigma \times \frac{3\pi}{8} \left[ 4(1-r^2 a^{-2}) + \frac{c}{d} \right] \]

\[ c_4 = \frac{1}{2} \rho \rho \rho h \left( u u \cos(\theta) \right) \]

Performing a force balance in the X-Z plane to obtain the jet trajectory using the coordinate system shown in the Fig. 1, the equation for jet curvature angle with respect to time will be [1]:

\[ \frac{d\theta}{dt} = \frac{(F_{\text{area}} - F_{\text{shear}}) \cos(\theta)}{\rho \rho \pi a b h v} \]  

where:

\[ F_{\text{shear}} = F_1 - F_2 = \pi a b \rho \rho \left( u u \cos(\theta) \right) \]

\[ F_{\text{area}} = C_a a h \rho \rho \left( u u \cos(\theta) \right) \]

The velocity of the center of mass of the element can be written in the form of:

\[ U_x = v_x \sin(\theta) \]

\[ U_y = v_y \cos(\theta) \]

### 2.2 Mass Reduction

As the jet moves along its trajectory, droplets and fragments strip from the liquid column. The rate of mass stripping at different heights of the liquid column strongly depends on the air flow conditions and the breakup regime. Since the current model studies the deformation and trajectory of the liquid column, we limit the scope of the present study to bag/shear and shear breakup regimes. Thus, according to Mazallon et al. [4], the mass stripping in our model starts after meeting the following criteria:

\[ \text{We}_{critical} = 60 \times \text{We}_{local} \]  

where

\[ \text{We}_{local} = \frac{\rho \rho \rho u u^2 2a}{\sigma} \]  

The equation for the mass shedding from the cylindrical element of our model would be [1]:

\[ M_{\text{shed}} = \frac{3}{4} \left( \pi d \right)^{\frac{3}{2}} \rho_l \rho \frac{1}{t} G H \rho u u \times R M t \times \]  

where,

\[ R_M = h \rho_l \rho \rho r \frac{3h}{4} \rho \rho r \frac{4r}{r} \]

\[ t = \frac{d\sqrt{\rho_l \rho \rho r}}{u u} \]

This equation has been offered by Rangers and Nicholls [6] and the term \( t \) has been added to enforce a smooth increase in the mass shedding from the shedding starting point. The term \( t \) in the above equation is the time elapsed from the start of mass stripping.
2.3 Drag Coefficient

Drag coefficient, \( C_D \), plays an important role in the prediction of the jet trajectory. To use the correct value for the Drag coefficient in equation (8), \( C_D \) was calculated numerically for \( e=1, 2, 4 \) at air Reynolds numbers of 150-8000. The calculations were carried out using FLUENT Computational Fluid Dynamics (CFD) code [6]. The details and validation of the numerical simulations can be found in [1,7]. Following are the equations for the curves fitted to the data points:

\[
e=1:
\]
\[
C_D = (-3*10^{-12})Re^{2}+(5*10^{8})Re^{0.5} - 0.0002Re + 1
\]
\[
e=2:
\]
\[
C_D = (-4*10^{-16})Re^{2}+(9*10^{12})Re^{0.5} - (9*10^{8})Re + 0.0004Re + 1.96
\]
\[
e=4:
\]
\[
C_D = 7.3979Re^{0.2166}
\]

The value of the drag coefficient at each time step in the model will be computed by first calculating the Reynolds number and then interpolating linearly between the above curves according to the values of the instantaneous aspect ratio, \( e \). It should be noted that the above curves are only valid within the mentioned range of calculations.

Equations (5), (6) are integrated simultaneously using 4th order Runge-Kutta method with time steps smaller than 10^{-6} sec. Equation (5) solves for the deformation of the cross section and equation (6) solves for the \( \theta \), the trajectory angle. Finally, equations (9) and (10) are integrated to determine the \( X \) and \( Z \) coordinates of the center of mass of the elliptical element versus time. The following transformations should be applied to transfer the coordinates from the center of mass of the element to the upper boundary of the spray plume.

\[
X_{u.b.} = r_0 + X_{c.m.} - b\cos(\theta)
\]
\[
Z_{u.b.} = Z_{c.m.} + b\sin(\theta)
\]

The subscripts “u.b.” and “c.m.” correspond to upper boundary and center of mass respectively.

3. RESULTS AND DISCUSSION

In this section we discuss the effects of liquid-to-jet momentum ratio, pressure, air Weber number, nozzle diameter and mass reduction on the jet trajectory.

3.1 Effect of Momentum Ratio

Figure 3 demonstrates the calculated jet shape for three different cases. The jet trajectories for the same test cases are plotted in Fig. 4 in a non-dimensional coordinate system along with experiments of Wu et al. (1997) [8]. As the momentum ratio \( q \) increases, the jet tends to penetrate more into the gas stream as stated before by many literatures. However, we will see that the value of \( q \) is not the only parameter in determination of the jet path. Figure 5 compares the calculated trajectory with experiment of We et al. [8] and empirical correlations offered by Chen et al. (1997) [9], Becker and Hassa (2002) [10], Wu et al. (1997) [8] and Wotel et al. (1991) [11].

3.2 Effect of Pressure

Figure 6 plots the calculated trajectories and compares them with the high pressure experimental values of Rachner et al. [11]. The pressure is ranging from 5.8 to 8.7 for the curves and the notations used in Fig. 6 are taken from [11]. The details of the test conditions are summarized in table 1. As the figure shows, the calculations slightly underestimate the penetration. This could be due to the fact that at higher pressures, the mass reduction equation used in the current model underestimates the mass stripped from the liquid column. Another probable reason is that at higher pressures the calculated drag coefficients may be smaller than the values calculated for normal pressures. Figure 7 compares the trajectory calculated by the present model for a case of high pressure (5.8 bar) with the same empirical correlations. The two empirical correlations used in the Fig. 7 offered by Elshamy and Jeng. [13] and Becker and Hassa [10] were specifically derived for high pressures.

3.3 Effect of Weber number

Figure 8 shows the variation in the jet trajectory as the Weber number increases at a constant liquid-to-air momentum ratio. The term ‘No stripping’ for the case of \( We=8.6 \) refers to the fact that the local Weber number for this test case does not pass the criteria of equation (11). The term ‘transition’ refers to the fact that the local Weber number exceeds the limit somewhere on the jet trajectory and the stripping starts some time after the injection point. The term ‘Mass Stripping’ refers to the fact that the jet is in the shear breakup and the local Weber number is initially above the criteria and thus, mass is stripped off the column from the injection point. The jet deformation gets larger as the air Weber number grows leading to increase in the frontal area used in the equations of motions of the jet and finally leading to lower penetrations as the Fig. 8 offers. This fact is also shown in Fig. 9 where the penetration curves for the very same test cases are plotted using the empirical correlation offered by Elshamy and Jeng. [13] which counts for the air Weber number as well as the pressure and momentum ratio. Although the trajectories are not exactly the same, they both demonstrate the same behavior while increasing the Weber number.

3.4 Effect of the Nozzle Diameter

Figure 10 compares the jet penetrations for three test cases with different nozzle diameters. The momentum ratio and air Weber numbers are held constant to merely investigate the effect of \( d \). An 11% difference in the penetration height can be observed when the diameter is increased from 0.5 mm to 2 mm. The air and jet velocities were varied in Fig. 10 to keep
the \( q \) and \( We \) constant. Figure 11 plots the trajectories for three test cases where the air and jet velocities were kept constant and the nozzle diameter was varied again. Since the jet and air velocities are constant, the momentum ratio, \( q \), remains constant. However, the Weber number grows larger with increasing the nozzle diameter. As the Weber number grows, the jet tends to bend faster due to rapid deformation as discussed in the previous section (Fig. 8 & 9).

3.5 Mass Reduction

Figure 12 plots the local Weber number versus the non dimensional time for three test cases with different air velocities. The jet velocity is kept constant for the test cases. Figure 13 plots the ratio of the mass stripped versus the non dimensional time. As the figure 12 depicts, for the first curve with air velocity of 50 m/s, the local Weber number does not exceed the stripping limit line. Thus, no mass would be removed from the column in the present model as shown by the corresponding curve in Fig. 13. For the second case, the local Weber number exceeds the limit at the non dimensional time of almost 0.6 and as the Fig. 13 shows, mass stripping would start after this point smoothly from zero. For the third test case with air velocity of 90 m/s, the mass is removed from the column from the injection point since its local Weber number starts from the value of nearly 70 m/s which is initially above the limit.

4. CONCLUSION

The presented model made it possible to investigate the effect of different parameters on the jet penetration in a fast manner. Good agreement of the results with available experiments confirms that the used model provides good approximation of the real conditions.

The results show that the momentum ratio is not the only effective parameter in the calculation of the jet trajectory and deformation in the gas stream. The jet path is function of the jet deformation, drag force, liquid and gas properties, breakup regime and other parameters. Since the change in any of the above parameters affects the others, they all should be considered simultaneously to calculate the jet penetration theoretically. As investigated in this article, the jet deformation is greatly affected by liquid and gas properties, drag force and mass reduction from the column. The mass reduction is function of Weber number (that determines the breakup type) and the pressure. The nozzle diameter can also slightly affect the penetration as investigated.

5. NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>angle between trajectory and vertical plane</td>
</tr>
<tr>
<td>( W )</td>
<td>work</td>
</tr>
<tr>
<td>( p )</td>
<td>pressure</td>
</tr>
<tr>
<td>( F )</td>
<td>force</td>
</tr>
<tr>
<td>( m )</td>
<td>mass</td>
</tr>
<tr>
<td>( u )</td>
<td>velocity</td>
</tr>
<tr>
<td>( \rho )</td>
<td>density</td>
</tr>
<tr>
<td>( \mu )</td>
<td>viscosity</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>surface tension coefficient</td>
</tr>
<tr>
<td>( C_D )</td>
<td>drag coefficient</td>
</tr>
<tr>
<td>( q )</td>
<td>Liquid/gas momentum ratio</td>
</tr>
<tr>
<td>( We )</td>
<td>Weber number</td>
</tr>
<tr>
<td>( Re )</td>
<td>Reynolds number</td>
</tr>
</tbody>
</table>

Subscripts

\( g \) gas
\( l \) liquid
\( rel \) relative
\( j \) jet
\( u.b. \) upper boundary
\( c.m. \) center of mass
\( v \) viscous
\( s \) surface
\( eq \) equal

6. REFERENCES

7. FLUENT User Manual, Fluent Inc.
Table 1. Test condition for high pressure cases from Rachner et al. [26].

<table>
<thead>
<tr>
<th>Test Case name</th>
<th>Air pressure [bar]</th>
<th>Air Temp. [K]</th>
<th>Air Density [kg/m³]</th>
<th>Air Velocity [m/s]</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>q18</td>
<td>5.8</td>
<td>280</td>
<td>7.19</td>
<td>100</td>
<td>18</td>
</tr>
<tr>
<td>p9</td>
<td>8.7</td>
<td>280</td>
<td>10.78</td>
<td>100</td>
<td>6</td>
</tr>
<tr>
<td>u75</td>
<td>5.9</td>
<td>285</td>
<td>7.18</td>
<td>75</td>
<td>6</td>
</tr>
<tr>
<td>u75q2</td>
<td>5.9</td>
<td>285</td>
<td>7.18</td>
<td>75</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 1. Schematic of the jet element movement along the trajectory and the aerodynamic force.

Figure 2. Analogy between an oscillating two-dimensional drop and a forced mass-spring system.

Figure 3. Calculated Jet shapes for three cases with different momentum ratios.
Figure 4. Comparison of Calculated Trajectories with experiments for cases with different momentum ratios.

Figure 5. Comparison of the present calculation with some empirical correlations.

Figure 6. Calculations for high pressure test cases and comparison with experiments of Rachner et al. [11]

Figure 7. Comparison of the present calculation with some empirical correlations for a high pressure case.

Figure 8. Effect of Weber number on the trajectory at a constant momentum ratio using present model.

Figure 9. Effect of Weber number on the trajectory at constant momentum ratio using the empirical correlation of Elshamy and Jeng [13].
Figure 10. Effect of the nozzle diameter at constant momentum ratio and Weber number.

Figure 11. Effect of the nozzle diameter at constant air and jet velocities.

Figure 12. Mass reduction from the column at different Weber numbers.