NUMERICAL FLOW SIMULATION ABOUT DROPLETS AND BUBBLES USING VOLUME CONSERVATIVE LEVEL SET METHOD

Koji Morinishi 1, Yoshitaka Wada 2 and Masanori Obata 3

1Kyoto Institute of Technology, morinishi@kit.ac.jp
2Kyoto Institute of Technology, yoshitaka@kit.ac.jp
3Kyoto Institute of Technology, masanori@kit.ac.jp

ABSTRACT This paper describes numerical flow simulations of incompressible gas-liquid two-phase flows using an artificial compressibility method and a forced volume (mass) conservative level set method. The artificial compressibility method is adopted to compute the variable density incompressible flows easily. The level set method is adopted to compute the gas-liquid two-phase flows with surface tension accurately. The conventional level set methods, however, have a common disadvantage in the volume (mass) conservation when the interfaces experience severe stretching or tearing. To overcome the deficiency, we propose the level set method with which the volume (mass) of each droplet or bubble is independently conserved, even if the droplet or bubble experience the coalescence or breakup. Validation of the method is carried out for two-dimensional and fully three-dimensional flow simulations of water droplets and air bubbles. The maximum volume fluctuation of the droplets and bubbles produced with the present method is less than 0.1%. The numerical results obtained are well compared with available experiments and other numerical results. It is confirmed that the present method is effective to simulate the incompressible gas-liquid two-phase flows which include coalescence or breakup of the droplets or bubbles.

Keywords: Computational Fluid Dynamics, Level Set Method, Droplet, Bubble

1. INTRODUCTION

Several numerical methods, including boundary integral method [1], front-tracking method [2], volume-of-fluid (VOF) method [3], constrained interpolation profile (CIP) method [4], phase field method [5], second gradient method [6], and level set method [7], have been proposed for simulating the gas-liquid two-phase flows. All of these methods have their advantages and disadvantages. Among them, the level set methods have been intensively studied for the past decade [8-11], because the method can treat three dimensional severe topological changes of the gas-liquid interfaces under surface tension easily.

The conventional level set methods, however, have a common disadvantage in the volume (mass) conservation (loss or gain of volume) when the interfaces experience severe stretching or tearing. In order to overcome the disadvantage, Sussman et al. [10] combined the level set method with adaptive mesh refinement (AMR) [12]. The AMR method enabled them to increase the grid resolution at regions near the free surface, while 10 to 20% loss of volume was still observed in their numerical results.

Sussman and Puckett [11] proposed a coupled level set/volume-of-fluid (CLSVOF) method for incompressible two-phase flows. The coupled method inherits abilities of volume conservative property from the volume-of-fluid method [13] and accurate treating surface tension from the level set method. Though the coupled method has advantages of both the methods, the simplicities of the level set method, however, may be lost in the coupled method.

In this paper, we propose a forced volume conservative level set method with which volume (mass) of each droplet and bubble can be independently conserved, even if the droplet and bubble experience the coalescence or breakup. In our method, the original volume of each droplet (or bubble) is distinguished from one another. Even if any loss or gain of the volume is produced in the computation, the level set function in each droplet is independently modified so that the correct volume is recovered. If two droplets merge into a droplet, their volume is also merged. If a droplet breaks down into two droplets, the original volume is also divided between the two droplets.

2. GOVERNING EQUATIONS

2.1 Level Set Function

The level set function \( \phi \) is typically defined as the signed normal distance from the interface between the gas and the liquid [7]. The level set function is usually positive in the liquid, negative in the gas, and zero at the interface. The convection of the interface is given by

\[
\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0
\]  

(1)

where \( \mathbf{u} \) is the fluid velocity.

The unit normal vector \( \mathbf{n} \) and the curvature of the interface \( \kappa \) can be obtained with the level set function \( \phi \) as

\[
\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}
\]

(2)

\[
\kappa = \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right)
\]

(3)
The density $\rho$ and the viscosity $\mu$ in the incompressible gas-liquid two-phase flows are defined using the level set function $\phi$ as

$$\rho(\phi) = \begin{cases} 
\rho_l & \phi \geq \alpha \\
\bar{\rho} + \Delta \rho \sin(\pi \phi/2\alpha) & \phi < \alpha \\
\rho_g & \phi \leq -\alpha 
\end{cases}$$

(4)

$$\mu(\phi) = \begin{cases} 
\mu_l & \phi \geq \alpha \\
\bar{\mu} + \Delta \mu \sin(\pi \phi/2\alpha) & \phi < \alpha \\
\mu_g & \phi \leq -\alpha 
\end{cases}$$

(5)

where $\rho_l$ and $\mu_l$ are the density and the viscosity of the liquid, and $\rho_g$ and $\mu_g$ are those of the gas, respectively. Here the mean value and difference are defined as

$$\bar{\rho} = (\rho_l + \rho_g)/2, \ \Delta \rho = (\rho_l - \rho_g)/2$$

(6)

$$\bar{\mu} = (\mu_l + \mu_g)/2, \ \Delta \mu = (\mu_l - \mu_g)/2$$

(7)

In Eqs. (4) and (5), a thickness of $2\alpha$ is given to the interface in order to prevent numerical instability which may be caused by the sharp change of flow quantities at the gas-liquid interfaces.

### 2.2 Navier-Stokes Equations

The governing equations for incompressible two-phase flow can be written in non-dimensional form as [3, 10]

$$\nabla \cdot \mathbf{u} = 0$$

(8)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho(\phi)} \nabla p = \frac{1}{Fr} \mathbf{g} + \frac{1}{Re} \nabla \mu(\phi) \mathbf{D} + \frac{1}{We} \kappa(\phi) \delta(\phi) \mathbf{n}$$

(9)

where $p$ is the pressure, $\mathbf{g}$ is the unit gravitational vector, $\mathbf{D}$ is the rate of deformation tensor which is given by

$$\mathbf{D} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

(10)

The modified delta function $\delta(\phi)$ is defined as

$$\delta(\phi) = \begin{cases} 
\frac{1}{2\alpha} [1 + \cos(\pi \phi/\alpha)] & \phi < \alpha \\
0 & \phi \geq \alpha 
\end{cases}$$

(11)

In this paper, the dimensionless parameters, namely Reynolds number $Re$, Weber number $We$, and Froude number $Fr$ are defined as

$$Re = \frac{\rho UL}{\mu_l}$$

(12)

$$We = \frac{\rho U^2 L}{\gamma}$$

(13)

$$Fr = \frac{U^2}{gL}$$

(14)

where $U$ is the reference velocity, $L$ is the reference length, $\gamma$ is the surface tension coefficient, and $g$ is the acceleration of gravity. Hereafter $\sqrt{gL}$ is used for the reference velocity $U$ and unit Froude number ($Fr = 1$) is specified, unless the specific note is provided.

### 2.3 Artificial Compressibility Method

In order to compute the variable density incompressible flows easily, the artificial compressibility method [14] is adopted. The resulting incompressible Navier-Stokes equations may be written in a conservative form as

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{h} = \frac{1}{Fr} \mathbf{G}$$

(15)

where dependent variable and flux vectors are defined as

$$\mathbf{q} = \begin{pmatrix} p \\ u \\ v \\ w \end{pmatrix}, \ \mathbf{h} = \begin{pmatrix} 0 \\ u \\ v \\ w \end{pmatrix}, \ \mathbf{F}_x = \begin{pmatrix} \beta u \\ u^2 + \frac{1}{[\rho]} p \\ uv \\ uw \end{pmatrix}$$

$$\mathbf{F}_y = \begin{pmatrix} \beta v \\ uv \\ v^2 + \frac{1}{[\rho]} p \\ vw \end{pmatrix}, \ \mathbf{F}_z = \begin{pmatrix} \beta w \\ uw \\ vw \\ w^2 + \frac{1}{[\rho]} p \end{pmatrix}$$

(16)

$$\mathbf{G} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{R}_x = \begin{pmatrix} 0 \\ \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{pmatrix}, \ \mathbf{R}_y = \begin{pmatrix} \sigma_{yy} \\ \sigma_{yx} \\ \sigma_{yz} \\ \sigma_{zy} \end{pmatrix}$$
Here \( u \), \( v \), and \( w \) are the velocity components, \( n_x \), \( n_y \), and \( n_z \) are the unit normal vector components on the interface, \( \tau \) is a pseudo-time and \( \beta \) is an artificial compressibility parameter, respectively. The components of viscous stress tensor \( \sigma \) are defined as

\[
\sigma_{xx} = 2\mu(\phi) \frac{\partial u}{\partial x}, \quad \sigma_{xy} = \mu(\phi) \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right),
\]

\[
\sigma_{yx} = 2\mu(\phi) \frac{\partial v}{\partial y}, \quad \sigma_{yz} = \mu(\phi) \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right),
\]

\[
\sigma_{zx} = 2\mu(\phi) \frac{\partial w}{\partial z}, \quad \sigma_{zz} = \mu(\phi) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right).
\]

It should be noticed that the density in the square brackets \( \{ \rho \} \) in Eq. (16) is the locally constant density and is not affected by the difference operators in Eq. (15).

To advance the solution of the convective equation of the level set function simultaneously, a difference term of level set function coupled with the solution of the Navier-Stokes equations is introduced to the convective equation as

\[
\frac{\partial \phi}{\partial \tau} + \frac{\partial \phi}{\partial t} + \frac{\partial u \phi}{\partial x} + \frac{\partial v \phi}{\partial y} + \frac{\partial w \phi}{\partial z} = 0
\]

### 3. NUMERICAL PROCEDURE

The solutions of the Navier-Stokes equations (15) and convective equation of the level set function (18) are obtained using Cartesian grids. All discrete variables, namely, the velocity \( \mathbf{u}_{i,j,k} \), the pressure \( p_{i,j,k} \), and the level set function \( \phi_{i,j,k} \) are located at cell centers, where subscripts \( i \), \( j \), and \( k \) denote the computational cell indexes for \( x \), \( y \), and \( z \) directions, respectively.

A amount of sub-iterations in the pseudo-time level are carried out at every physical time step so that unsteady incompressible flow solution is obtained using the artificial compressibility method. Here the index of sub-iteration step is denoted with a superscript \( m \) and the index of physical time step is denoted with a superscript \( n \). The solution of next physical time step \( \mathbf{q}^{n+1} \) is obtained as

\[
\mathbf{q}^{n+1} = \mathbf{q}^{n+1,m+1} \text{ if } |\mathbf{q}^{n+1,m+1} - \mathbf{q}^{n+1,m}| < \varepsilon.
\]

where \( \varepsilon \) is a positive small number of tolerable order.

### 3.1 Estimation of Convective Terms

The convective terms of the governing equations are estimated by an upwind difference with the third order accurate weighted essentially non-oscillatory (WENO) reconstruction [15]. For example, the convective term in \( x \)-direction of the Navier-Stokes equations is estimated as

\[
\left. \frac{\partial \mathbf{F}_x}{\partial x} \right|_{i,j,k} = \frac{\mathbf{F}_{x,i+1/2,j,k} - \mathbf{F}_{x,i-1/2,j,k}}{\Delta x} \tag{20}
\]

where \( \Delta x \) is the grid spacing in \( x \)-direction. The numerical flux at the right cell interface \( \mathbf{F}_{x,i+1/2,j,k} \) is obtained as

\[
\mathbf{F}_{x,i+1/2,j,k} = \frac{1}{2} \left[ \mathbf{F}_x^r + \mathbf{F}_x^l \right] \left[ (\mathbf{q}^r - \mathbf{q}^l) \right]_{i+1/2,j,k} \tag{21}
\]

where \( \mathbf{A}_x \) is the flux Jacobian matrix defined as

\[
\mathbf{A}_x = \frac{\partial \mathbf{F}_x}{\partial \mathbf{q}} \tag{22}
\]

Equation (20) may be the third order accurate, if \( \mathbf{q}^r_{i+1/2,j,k} \) and \( \mathbf{q}^l_{i+1/2,j,k} \) are obtained with the following WENO reconstruction.

\[
\mathbf{q}^r_{i+1/2,j,k} = \omega_0^r \mathbf{q}_{i+1/2,j,k} + \omega_1^r \mathbf{q}_{i+1,j,k} \tag{23}
\]

\[
\mathbf{q}^l_{i+1/2,j,k} = \omega_0^l \mathbf{q}_{i+1/2,j,k} + \omega_1^l \mathbf{q}_{i-1,j,k}
\]

where

\[
\omega_0^r = \frac{3}{2} \mathbf{q}_{i+1,j,k} - \frac{1}{2} \mathbf{q}_{i+2,j,k}
\]

\[
\mathbf{q}^{i+1/2}_{i+1,j,k} = \frac{1}{2} (\mathbf{q}_{i+1,j,k} + \mathbf{q}_{i,j,k}) \tag{24}
\]

\[
\mathbf{q}^{i+1/2}_{i+1,j,k} = \frac{3}{2} \mathbf{q}_{i,j,k} - \frac{1}{2} \mathbf{q}_{i-1,j,k}
\]

and the weights are defined as

\[
\omega_0^r = \frac{\lambda_0}{\lambda_0 + \lambda_1}, \quad \omega_1^r = \frac{\lambda_1}{\lambda_0 + \lambda_1}
\]

\[
\omega_0^l = \frac{\lambda_0}{\lambda_0 + \lambda_{-1}}, \quad \omega_1^l = \frac{\lambda_{-1}}{\lambda_0 + \lambda_{-1}} \tag{25}
\]

with
\[ \lambda_i = \frac{1}{3} \left( q_{i+2,j,k} - q_{i+1,j,k} \right)^2 + \varepsilon \]

\[ \lambda_j = \frac{2}{3} \left( q_{i+1,j,k} - q_{i,j,k} \right)^2 + \varepsilon \]

\[ \lambda_{-j} = \frac{1}{3} \left( q_{i,j,k} - q_{i-1,j,k} \right)^2 + \varepsilon \] (26)

Here \( \varepsilon \) is a positive small number to avoid the denominator becoming zero.

The convective terms of the level set equation (18) is also estimated with similar manner.

### 3.2 Estimation of Viscous Terms

The viscous terms of the Navier-Stokes equation (15) are estimated with the second order accurate central difference approximations. For example, the viscous term in \( x \)-direction is estimated as

\[ \frac{\partial R_x}{\partial x} \bigg|_{i,j,k} = \frac{1}{\Delta x} \begin{pmatrix} 0 \\ \frac{\sigma_{x+i+1/2,j,k} - \sigma_{x-i+1/2,j,k}}{\Delta x} \\ \frac{\sigma_{y+i+1/2,j,k} - \sigma_{y-i+1/2,j,k}}{\Delta x} \\ \frac{\sigma_{z+i+1/2,j,k} - \sigma_{z-i+1/2,j,k}}{\Delta x} \end{pmatrix} \] (27)

The viscous stress at the cell interface is also estimated with the central difference approximations as

\[ \sigma_{x+i+1/2,j,k} = 2\mu_{i+1/2,j,k} \frac{u_{i+1,j,k} - u_{i,j,k}}{\Delta x} \] (28)

\[ \sigma_{y+i+1/2,j,k} = \mu_{i+1/2,j,k} \left( \frac{v_{i+1,j,k} - v_{i,j,k}}{\Delta x} \right) \] (29)

\[ \mu_{i+1/2,j,k} = \frac{1}{2} \left( \mu_{i+1,j,k} + \mu_{i,j,k} \right) \] (30)

### 3.3 Estimation of Physical Time Derivative

The physical time derivatives are estimated with the second order accurate three-point backward difference formula. For example, the physical time derivatives of the Navier-Stokes equations (15) are estimated as

\[ \frac{\partial q^{n+1}}{\partial t} \bigg|_{i,j,k} = \frac{3q^{n+1} - 4q^n + q^{n-1}}{2\Delta t} \] (31)

where \( \Delta t \) denotes the physical time step.

### 3.4 Pseudo-time Stepping Method

The sub-iteration in the pseudo-time level is carried out at every physical time step using the lower-upper symmetric Gauss Seidel (LU-SGS) implicit method [16]. If the backward Euler implicit difference approximation is applied to the pseudo-time derivative of the Navier-Stokes equations (15), we have:

\[ \begin{bmatrix} \mathbf{I}' + \frac{\partial}{\partial x} \mathbf{A}_x + \frac{\partial}{\partial y} \mathbf{A}_y + \frac{\partial}{\partial z} \mathbf{A}_z \end{bmatrix} \Delta \mathbf{q} = \mathbf{RHS}^{n+1,m} \] (32)

where \( \mathbf{RHS} \) is the estimated sum of each term in Eq. (15) except the pseudo-time derivative term and \( \mathbf{I}' \) is a diagonal matrix defined by

\[ \mathbf{I}' = \text{diag} \left[ \frac{1}{\Delta t}, \frac{1}{2\Delta t}, \frac{1}{2\Delta t} \right] \] (33)

The correction \( \Delta \mathbf{q} \) is defined as

\[ \Delta \mathbf{q} = \mathbf{q}^{n+1,m+1} - \mathbf{q}^{n+1,m} \] (34)

Introducing an approximate flux Jacobian splitting and applying the first order upwind difference to the left hand side of Eq. (32), the equation can be solved using the LU-SGS method as

\[ \Delta \mathbf{q}^*,_{i,j,k} = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{RHS}^{n+1,m} + \frac{1}{\Delta x} \mathbf{A}_x^\pm \Delta \mathbf{q}^*,_{i-1,j,k} \\
+ \frac{1}{\Delta y} \mathbf{A}_y^\pm \Delta \mathbf{q}^*,_{i,j-1,k} \\
+ \frac{1}{\Delta z} \mathbf{A}_z^\pm \Delta \mathbf{q}^*,_{i,j,k-1} \end{bmatrix} \] (35)

\[ \Delta \mathbf{q}^*,_{i,j,k} = \Delta \mathbf{q}^*,_{i,j,k} - \mathbf{B}^{-1} \begin{bmatrix} \frac{1}{\Delta x} \mathbf{A}_x^\pm \Delta \mathbf{q}^*,_{i+1,j,k} \\
+ \frac{1}{\Delta y} \mathbf{A}_y^\pm \Delta \mathbf{q}^*,_{i,j+1,k} \\
+ \frac{1}{\Delta z} \mathbf{A}_z^\pm \Delta \mathbf{q}^*,_{i,j,k+1} \end{bmatrix} \] (36)

where the flux Jacobian approximately split, for example, \( \mathbf{A}_x^\pm \) can be defined as

\[ \mathbf{A}_x^\pm = \frac{1}{2} \left( \mathbf{A}_x \pm b_x \mathbf{I} \right) \] (37)

Here \( \mathbf{I} \) is the identity matrix and \( b_x \) may be given by

\[ b_x = u + \sqrt{\frac{u^2 + \beta}{\rho}} \frac{2\mu}{Re \rho \Delta x} \] (38)
The diagonal matrix $B$ is obtained as

$$B = I' + \left( \frac{b_x}{\Delta x} + \frac{b_y}{\Delta y} + \frac{b_z}{\Delta z} \right) I$$  \hspace{1cm} (39)$$

The solution in the sub-iteration is updated with the correction as

$$q^{n+1,m+1} = q^{n+1,m} + \Delta q$$  \hspace{1cm} (40)$$

The level set function is also updated with similar manner through the sub-iteration coupled with the Navier-Stokes equations and the re-distance (reinitializing) iteration [9].

4. FORCED VOLUME CONSERVATION

4.1 Forced Conservation

Since the level set function $\phi$ is the signed normal distance from the interface between the gas and the liquid, and is positive in the liquid and negative in the gas, the volume of a droplet whose index is $k$ can be obtained by

$$V_k = \int_{\Omega_k} H(\phi) d\Omega$$  \hspace{1cm} (41)$$

where $\Omega_k$ is the domain occupied by the droplet $k$ and $H(\phi)$ is a smoothed Heaviside function defined as

$$H(\phi)=\begin{cases} 0 & \phi \leq -\alpha \\ \frac{1}{2} \left[ 1 + \frac{\phi}{\alpha} + \frac{1}{\pi} \sin(\pi \phi/\alpha) \right] & |\phi| < \alpha \\ 1 & \phi \geq \alpha \end{cases}$$  \hspace{1cm} (42)$$

If the initial volume of the droplet is given by $V_k(0)$ and the volume at time $t$ by $V_k(t)$, the lost volume can be defined by

$$\Delta V_k = V_k(0) - V_k(t)$$  \hspace{1cm} (43)$$

The lost volume can be approximately recovered if the level set function $\phi$ in the domain $\Omega_k$ is modified as

$$\phi_{new} = \phi_{old} + \Delta \phi_k \text{ for } x \in \Omega_k$$  \hspace{1cm} (44)$$

where the collection $\Delta \phi_k$ is obtained by

$$\Delta \phi_k = \frac{\Delta V_k}{S_k}$$  \hspace{1cm} (45)$$

Here $S_k$ is the surface area of droplet $k$ obtained by

$$S_k = \int_{\Omega_k} \delta(\phi) d\Omega$$  \hspace{1cm} (46)$$

where $\delta(\phi)$ is the modified delta function which is given by Eq. (11).

4.2 Identification

Since each droplet may move every time to time, or may encounter coalescence or breakup, we must identify the root of each droplet to independently conserve its volume. In this subsection, for simplicity, we explain our method for a two dimensional space, while the extension to a three dimensional space is straightforward.

The domain occupied by each droplet $\Omega_k$ is obtained with the following search process.

(1) First a flag integer, which is initialized with zero, is assigned to each computational cell $(i,j)$.

(2) We start to sweep the computational domain from left lower corner to right upper corner, to search a computational cell whose flag integer is zero and level set function is negative. If we find the cell at which both the two conditions are satisfied, the cell must be occupied by a droplet, a serial number, that is, the index number of the droplet $k$ is assigned to the flag integer of the cell and the cell indexes are added as a new member to a cell list of the droplet $k$.

Then we move to the following step (3). If we find no more cell out, we stop this search process.

(3) Around each of the new member cells,

$$(i,j) \in (I - 1 \leq i \leq I + 1, J - 1 \leq j \leq J + 1)$$  \hspace{1cm} (47)$$

we search cells whose flag integer is zero and level set function is less than $\alpha/2$. If we find the cell at which both the two conditions are satisfied, since the cell must be occupied by a droplet, the same serial number $k$ is assigned to the flag integer of the cell and the cell indexes are added as a new member to the cell list of the droplet $k$.

(4) Repeat the above step (3) until all cells occupied by the droplet $k$ are found out. If we find no more cell, come back to the step (2) and continue to sweep the rest of computational domain.

After the above search process, we find all the domains occupied by the droplets as shown in Fig. 1.

Next we must identify the domain $\Omega_{k}^{n+1}$ occupied by
a droplet \( k' \) at the present time step \( t^{n+1} \) with the domain \( \Omega_k^n \) occupied by a droplet \( k \) at the previous time step \( t^n \).

The identification is carried out by comparing the cell serial number of the flag integer at the previous time step \( t^n \) with that at the present time step \( t^{n+1} \) as shown in Fig. 1. Though there are three droplets at both time steps, they are not necessarily identical between \( t^n \) and \( t^{n+1} \). It should be notified that those figures are exaggeratedly drawn to understand our method easily. Such abrupt topological changes of droplets do not occur through the small time step in a practical computation.

Some cells occupied by \( \Omega_1^n \) at \( t^n \) are occupied by \( \Omega_1^{n+1} \) at \( t^{n+1} \) and some cells occupied by \( \Omega_2^n \) are also occupied by \( \Omega_2^{n+1} \). On the other hand, a part of cells occupied by \( \Omega_3^n \) at \( t^n \) are occupied by \( \Omega_2^{n+1} \) at \( t^{n+1} \) and other part of cells are occupied by \( \Omega_3^{n+1} \). Thus we find that the two droplets \( \Omega_1^n \) and \( \Omega_2^n \) at the time step \( t^n \) merge into the droplet \( \Omega_1^{n+1} \) at the time step \( t^{n+1} \), and the droplet \( \Omega_3^n \) at the time step \( t^n \) collapse into the two droplets \( \Omega_2^{n+1} \) and \( \Omega_3^{n+1} \) at the time step \( t^{n+1} \).

Finally, we must balance each volume between \( \Omega_k^n \) and \( \Omega_k^{n+1} \) as

\[
V_{1}^{\text{new}} (t^{n+1}) = V_{1} (t^n) + V_{2} (t^n) \\
V_{2}^{\text{new}} (t^{n+1}) = V_{3} (t^n) \frac{V_{2}^{\text{old}} (t^{n+1})}{V_{2}^{\text{old}} (t^{n+1}) + V_{3}^{\text{old}} (t^{n+1})} \\
V_{3}^{\text{new}} (t^{n+1}) = V_{3} (t^n) \frac{V_{3}^{\text{old}} (t^{n+1})}{V_{2}^{\text{old}} (t^{n+1}) + V_{3}^{\text{old}} (t^{n+1})}
\]

where \( V_{2}^{\text{old}} (t^{n+1}) \) and \( V_{3}^{\text{old}} (t^{n+1}) \) are obtained before the balance using Eq. (41) for \( \Omega_2^{n+1} \) and \( \Omega_3^{n+1} \), respectively. The level set function \( \phi \) in the domain \( \Omega_k^{n+1} \) is modified independently using Eq. (44) and

\[
\Delta V_k = V_k^{\text{new}} (t^{n+1}) - V_k^{\text{old}} (t^{n+1})
\]

(49)

After the modification, the level set function in whole computational domain is reconstructed using the re-distance (reinitializing) iterations.

5. NUMERICAL RESULTS

5.1 Rise of Initially Spherical Gas Bubble

Rising of initially spherical gas bubble in a viscous liquid is computed for a simple test problem, using the liquid density of \( 8.76 \times 10^{-2} \text{ Kg/m}^3 \), the liquid viscosity of 0.118 \( \text{Pa s} \), and the surface tension coefficient of \( 3.22 \times 10^{-2} \text{ N/m} \). The diameter of the initial bubble is 1.2 \( \times 10^{-2} \text{ m} \), which corresponds to the bubble A in spherical cap bubbles of [17]. For the gas density and viscosity, which are not given in [17], standard values of the air at 20° C are used. The computation is carried out for a full three dimensional domain with \( 4 \times 4 \times 16 \), where the reference length is the diameter of the bubble, and the number of computational cells used is \( 64 \times 64 \times 256 \). Free-slip boundary conditions are used on all side of computational domain.

Figure 2 shows the spherical cap bubble obtained at a dimensionless time of 12. The rising bubble velocity predicted by the present method is about 0.21 \( \text{m/s} \) which can be compared well to the experimental prediction of 0.215 \( \text{m/s} \).
5.2 Merging of Two Viscous Gas Bubbles
Secondly, merging of two gas bubbles is simulated at a density ratio of 0.05, a viscosity ratio 0.04, a Reynolds number of 18.8, and a Weber number of 50. These parameters correspond to Fig. 15 in [11]. The reference length is the diameter of the bubbles. The dimensions of the computational domain used are $4 \times 4 \times 8$ and the number of computational cells is $64 \times 64 \times 128$. Figure 3 shows the two snapshots at dimensionless time of 12 and 20. The maximum volume error of each bubble produced with the present method is less than 0.1%. The results agree qualitatively with those of [11].

5.3 Pendant Water Drop
Finally, a pendant water drop falling from a top wall is simulated. We put a water droplet, whose initial volume is $0.512 \text{cm}^3$, on a top wall. The initial shape of the droplet is given by

\begin{equation}
    z = z_{\text{wall}} - C[1 + \cos(\pi r / L)] \quad r \leq L
\end{equation}

where $z_{\text{wall}}$ is the $z$ coordinate of the top wall, $C$ is a constant to adjust the initial volume, and $r = \sqrt{x^2 + y^2}$. We use a reference length $L$ of 0.8 cm and the standard conditions of water and air at 20°C.

Figure 4 shows two snapshots which correspond to just around the breaking up. Next breakup occurs near the root of neck and a long tail droplet falls after the first droplet. Breakdown of the long tail droplet, however, can not found, though the simulation is continued till a dimensionless time of 60. Figure 5 shows the results using same conditions but an artificial noise is introduced in the droplet neck at dimension less time of 40. The noise make breakup earlier and the long tail break down into three droplets. The results agree qualitatively with those of [18].

6. CONCLUSIONS
A forced volume conservative level set method has been developed. The maximum volume error produced with the method is less than 0.1%. The numerical results obtained are well compared with available experimental and numerical results. It is confirmed that the present method is effective to simulate the gas-liquid two-phase flows which include coalescence or breakup of the droplets or bubbles.

7. NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>flux Jacobian matrix</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>B</td>
<td>diagonal matrix, see Eq. (39)</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>b</td>
<td>spectral radii, see Eq. (38)</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>D</td>
<td>rate of deformation tensor</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>F</td>
<td>convective flux vector</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>Fr</td>
<td>Froude number</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>G</td>
<td>gravitational flux vector</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>g</td>
<td>unit gravitational vector</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>g</td>
<td>acceleration of gravity</td>
<td>$[\text{ms}^{-2}]$</td>
</tr>
<tr>
<td>H</td>
<td>smoothed Heaviside function</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>h</td>
<td>physical time difference vector</td>
<td>(dimensionless)</td>
</tr>
<tr>
<td>I</td>
<td>identity matrix</td>
<td>(dimensionless)</td>
</tr>
</tbody>
</table>
I' diagonal matrix, see Eq. (33) (dimensionless)
L reference length [m]
M surface tension flux vector (dimensionless)
n unit normal \((n_x, n_y, n_z)\) (dimensionless)
p pressure [Pa]
q pseudo-time difference vector (dimensionless)
R viscous flux vector (dimensionless)
Re Reynolds Number (dimensionless)
S surface area (dimensionless)
t physical time (dimensionless)
Δt physical time step (dimensionless)
U reference velocity [ms⁻¹]
u velocity \((u, v, w)\) (dimensionless)
V cubic volume (dimensionless)
We Weber number (dimensionless)
x Cartesian coordinate \((x, y, z)\) (dimensionless)
\(\Delta x\) grid spacing \((\Delta x, \Delta y, \Delta z)\) (dimensionless)
α half thickness of interface (dimensionless)
β artificial compressibility (dimensionless)
γ surface tension coefficient [Nm⁻¹]
δ modified delta function (dimensionless)
ε small positive tolerable number (dimensionless)
κ curvature of interface (dimensionless)
λ coefficient of WENO method (dimensionless)
μ viscosity coefficient [Pa s]
ρ density [kg m⁻³]
σ viscous stress (dimensionless)
τ pseudo-time (dimensionless)
ϕ level set function (dimensionless)
Ω domain occupied by droplet (dimensionless)
ω weight of WENO method (dimensionless)
s gas (cf. ρₙ) (dimensionless)
i grid index for x-direction (cf. \(u_{i,j,k}\)) (dimensionless)
k index of droplet (cf. \(V_k\)) (dimensionless)
l liquid (cf. \(ρ_l\)) (dimensionless)
l state of left side (cf. \(q_l^i\)) (dimensionless)
m index of pseudo-time step (dimensionless)
n index of physical time step (dimensionless)
r state of right side (cf. \(q_r^i\)) (dimensionless)
± positive or negative definite matrix (cf. \(A^+, A^-\))

8. REFERENCES