1. INTRODUCTION

Satellite drop formation is an interesting phenomenon in liquid atomization, mixing, and dispersion processes. To control the formation of satellite drops is essentially important, particularly for recent emerging technologies such as microfluidic devices: Inhibition of the formation of satellite drops is necessary for accurate ink jet printing [1] and active use of satellite drops is considered for the source of the mono-dispersed production of submicron emulsions [2]. In order to control effectively the formation of satellite drops, it is necessary to know the basic mechanisms of the formation. In this study, we numerically investigate the formation of satellite drops during the breakup of a liquid ligament suspended in another fluid.

Single fluid formulation of Navier-Stokes equations is used to simulate the motion of an initially stretched liquid ligament in another fluid [3]. A front-tracking / finite difference method [4] is applied to solving the governing equations and tracing the interfacial motion. We assume that the ligament motion and the fluid flow is axisymmetric with respect to the central axis of the ligament. Figure 1 shows the one of the computational results for the breakup of an initially stretched ligament. During the retraction of the ends of the ligament, bulbs form at both ends. The bulbs eventually break off from the rest of the ligament and they become two primary drops. Between the primary drop and the rest of the ligament, a thin thread forms and eventually breaks up into small satellite drops. The focus of this study is the satellite formation as shown in the frame at the upper right corner of Fig. 1.

The breakup of an infinite liquid ligament in another immiscible fluid has been extensively studied both theoretically [5-9] and experimentally [10]. The motion of an initially stretched finite liquid ligament has been studied theoretically, numerically, and experimentally by Stone and coworkers [11-13]. These previous studies have been very valuable in understanding the basic physics of the motion and the breakup of the liquid ligament suspended in another immiscible fluid. However, satellite formation is not particularly considered in these studies. Furthermore, the results only apply in the Stokes flow or the creeping flow limit, because the size of the ligament considered is very small and both the dispersed and the continuous phases are relatively viscous, and thus the Reynolds number is vanishingly small.

Satellite drop formation has been studied extensively in liquid-air systems, where ambient air does not affect the phenomenon. The study by Chaudhary and Maxworthy [14] is an example. The necking process just before pinch-off has recently considerable attention and is considered as the primary cause of the formation of satellite drops [15]. The interfacial shape near the pinch-off point is double cone and non-symmetry with respect to the pinch-off point: one is steep and the other is shallow. The
For liquid-liquid systems, the formation of satellite drops during the formation of drops from a nozzle has been visualized numerically by Xiaoguang [16] and experimentally by Milosevic and Longmire [17]. These studies give us insight into the satellite drop formation, but it is not sufficient to fully understand the mechanism. The necking process just before pinch-off in two immiscible liquids systems has been studied experimentally by Cohen et al. [18] and theoretically and numerically by Lister and Stone [19]. As well as liquid-air systems, the interfacial shape near the pinch-off point is double cone. Thus the satellite drop formation in liquid-liquid systems is similar to that in liquid-air systems. However, the shape of the double cone and the scaling of the thinning of the neck are different [18, 19].

Considering that the motion of a thin thread between primary drops is important to understand the formation of satellite drops, Notz and Basaran [20] has investigated numerically the motion of an initially stretched liquid ligament in air. Our study is the same direction as their study, but in liquid-liquid systems.

The rest of this paper is organized as follows: the problem statement and the governing equations are described with numerical method to solve the equations. The computational results for the motion of an initially stretched ligament in another liquid are reviewed based on our recent papers [3, 21]. The necking process near the pinch-off is then examined. Finally, the motion of the thin thread near the primary drops is discussed.

2. GOVERNING EQUATIONS AND NUMERICAL METHOD

2.1 Problem Statement and Mathematical Formulation

Figure 1 shows the schematic diagram of the problem. An initially stretched ligament of $\rho_0$ and $\mu_0$ is placed in another immiscible quiescent fluid of $\rho_1$ and $\mu_1$. The tip of the ligament starts to move toward the midsection due to high capillary pressure at the tip.

Two characteristic timescales of this phenomenon can be considered: viscous timescale, $t_v=\frac{a_0}{\mu_0}HD$, and inertial timescale, $t_I=(\frac{a_0}{\rho_0}^2/\sigma)^{1/2}$. Here, $\sigma$ is the interfacial tension and $a_0$ is the initial radius of the ligament. The ratio between the two timescales is the Ohnesorge number $[Oh_l=\frac{t_I}{t_v}=\frac{\mu_0(\rho_0\rho_1\sigma)^{1/2}}{\mu_0}]$, which is an index whether viscosity or inertia governs the phenomenon. The experimental study by Stone et al. [11] used 1 mm scale silicon oil drops with various viscosity (1-1000 P) suspended in Pale-4 Oil (50 P) and the order of the corresponding Ohnesorge number ranges from 1 to 1000. Thus the viscosity dominates the phenomenon. If we use a kerosene drop, whose viscosity is about $10^{-2}$ P, the order of the corresponding Ohnesorge number becomes $10^2$ and inertial effect comes into play. Hence full Navier-Stokes equations have to be solved when both viscous and inertial effects dominate the phenomenon. We deal with low Ohnesorge number range $[O(10^{-7}) < Oh_l < O(10^{-5})]$ in this study.

Both the ligament and the external fluid are assumed to be incompressible and Newtonian. It is also assumed that the system is isothermal and the interface is completely clean, so that there is no interfacial tension gradient. For a single fluid formulation, where two immiscible fluids are treated as a single fluid with different physical properties, the governing equations under these assumptions are the continuity equation, momentum equation, and the equations of state for density and viscosity:

\begin{equation}
\nabla \cdot u = 0,
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} \rho u + \nabla \cdot \rho uu = -\nabla P + \nabla \cdot \mu \left( \nabla u + \nabla u^T \right) + \int_0^\infty \sigma \kappa \delta \left( x - x_f \right) dA,
\end{equation}

\begin{equation}
\frac{D}{Dt} \rho = 0.
\end{equation}

Here, $\kappa$ is the curvature, $n$ the unit normal, $\delta$ the delta function, $x$ the position vector, and $A$ the interfacial area. Subscript $f$ stands for interface. Conventional symbols are used for other variables. The gravity is ignored in Eq. (2).

Two geometric simplifications are made to discretize Eqs. (1)-(3): When the flow and the motion of the ligament are assumed to be axisymmetric, cylindrical axisymmetric coordinates can be used. If the motion is also assumed to be symmetric with respect to the plane at the center of mass of the ligament, it is sufficient to take into account only a half of the either side of the ligament.

Full-slip wall conditions are applied to the left and the top of the domain in order to reduce the effect of boundary layer. Symmetric boundary is used for the right and the bottom of the domain. At $t=0$, the ligament and the external fluid are at rest, so that the all velocities in the domain are zero. The initial shape of the ligament is rectangular and the tip is flat. Two initial tip shapes, flat and semicircle, were tested and no significant difference was observed between two shapes.

2.2 Numerical Method

The governing equations are discretized with second order finite difference approximations for both time and space. The time dependent motion of the interface is computed using a Front Tracking method [4]. In this method, the interface is represented by sets of moving elements (fronts), immersed on the fixed computational grid where the pressure and the velocities are determined. A detailed description of the Front Tracking method is found in [22] and the method for a similar problem, axisymmetric jet formation from a nozzle and its breakup into drops, is described in [23].

3. COMPUTATIONAL RESULTS FOR LIGAMENT (REVIEW)
We have studied numerically the motion of an initially stretched liquid ligament suspended in another quiescent fluid [3, 21]. The results are briefly reviewed here. The problem statement and the numerical method used are the same as that described in Sec. 2. The numerical method was checked by a grid refinement test and a 128x2048 (1:16) grid mesh was determined as appropriate resolution for entire motion of the ligament.

When the initial aspect ratio of the ligament holds \( \frac{l_0}{a_0} = 120 \), three different modes were identified: (I) the ligament breaks up into several daughter drops (Fig. 1), (II) the ligament relaxes back to a spherical drop (Fig. 3), and (III) the ligament breaks up into two big drops (end-pinching [11]). Figure 4 shows the map of the mode on the diagram of \( \frac{\mu_c}{\mu_d} \) vs. \( Oh_d \). Mode (I) appears in small \( \frac{\mu_c}{\mu_d} \) and small \( Oh_d \). In this region, the viscosities of both the ligament and the external fluid are low. Thus the ligament is unstable and capillary wave instability clearly occurs. Even if the viscosity of the ligament is low \( (Oh_d < 10^{-3}) \), mode (II) is observed when the viscosity of the external fluid is high \( (\frac{\mu_c}{\mu_d} > 10) \). According to the linear stability analysis for two immiscible liquids systems [5], the viscous external fluid reduces the growth rate of disturbances. Thus the retraction of the ligament finishes before any capillary waves grow. The mode (II) also appears when the ligament is viscous \( (Oh_d > 0.5) \) and the external fluid is less viscous than the ligament. In this case, the ligament itself is stable and the growth of capillary waves on the ligament is suppressed. Mode (III) is a transition region between the modes (I) and (II).

To distinguish between the modes, we paid attention to the retraction velocity of the ligament and the growth rate of capillary waves. If the retraction overcomes the growth of capillary waves, the ligament relaxes back to a spherical drop before any capillary waves appear on the ligament. On the other hand, the ligament breaks up into daughter drops, if the growth of capillary waves is faster than the retraction. Hence two competing processes, the retraction of the ligament and the growth of capillary waves, determine the mode.

Figure 5 shows the time averaged retraction velocity, \( \bar{v}_r \), normalized by the characteristic velocity scale, \( v_{\phi} = \frac{a_0 l_0}{\rho \sigma} \). The retraction velocity increases with decreasing the Ohnesorge number and becomes constant when the Ohnesorge number is less than \( 10^{-2} \). For \( \frac{\mu_c}{\mu_d} = 10 \), the retraction velocity is low compared to the lower viscosity ratio of \( \frac{\mu_c}{\mu_d} = 1 \) and 0.1. Viscous external fluid
overcomes the interfacial tension, which promotes the retraction of the ligament.

Figure 6 shows the growth rate of the most unstable disturbance calculated by linear stability theory [5]. The growth rate is normalized by the characteristic timescale, \(t_\rho\). The trend is very similar to the retraction velocity as shown in Fig. 5.

We attempt to distinguish the modes based on the assumption of linear stability theory. When an infinitesimal disturbance of \(\delta_0\) grows exponentially with the rate \(\omega\) and the disturbance becomes as large as the radius of the ligament \(a_0\), the ligament is assumed to brake up. Thus the breakup time is obtained as

\[ \tau = \frac{1}{\omega} \ln \left( \frac{a_0}{\delta_0} \right). \]  

If the breakup time equals \(l_0/v_r\), which is the time when the ligament becomes a spherical drop, the ligament does not break. Hence the ligament breaks up into daughter drops when

\[ \omega \frac{1}{\ln \left( \frac{a_0}{\delta_0} \right)} < \frac{l_0}{v_r}, \]

and it does not break up when

\[ \omega \frac{1}{\ln \left( \frac{a_0}{\delta_0} \right)} > \frac{l_0}{v_r}. \]

Since the initial infinitesimal disturbance \(\delta_0\) varies with experimental conditions, it is difficult to predict the magnitude of \(\delta_0\). We therefore select \(\omega l_0/v_r\) as an index to distinguish the modes.

Figure 7 shows the map for the modes on the diagram of \(\mu_c/\mu_d\) and \(\omega l_0/v_r\). (I) daughter drops, and (II) single drop.

Fig. 7 Mode of the ligament motion on the diagram of \(\mu_c/\mu_d\) and \(\omega l_0/v_r\). (I) daughter drops, and (II) single drop.

4. RESULTS AND DISCUSSION

4.1 Pinch-Off Dynamics

Primary cause of the formation of satellite drops is non-symmetry interfacial shape with respect to the point at pinch-off [15]. It is thus important to know the necking process near pinch-off point. We investigate the necking process during the breakup of an initially stretched ligament.

Figure 8 shows the flow field in the region near the pinch-off will take place. The Ohnesorge number and the viscosity ratio is the same as those of Fig. 1 \((\text{Oh}_d=5.9 \times 10^{-2}, \mu_c/\mu_d=1)\). However, the resolution of the grid meshes is higher (256x4096), because the focus in this discussion is the local dynamics of the pinch-off point and the motion of the thin thread rather than entire motion of the ligament. The first pinch-off of the thin thread behind the primary drop at the end occurs at the ligament side [Fig. 8 (a)]. Close to the pinch-off, the motion of the interface is self-similar and the shape of the interface holds until the interface pinches [15]. This was checked by our early study for the same Ohnesorge number and the same viscosity ratio [24]. Due to the curvature change at the neck, the minimum pressure is observed inside the neck [Fig. 8 (b) and (c)]. The fluid inside the thread flows to the minimum point and the pinch off eventually occurs due to mass conservation.

The scaling is examined for the radial location of the neck (Fig. 9). The time \(t^*\) in Fig. 9 is the breakup time, so that \(t^*-\tau\) means the time from the pinch off and becomes zero at pinch-off. The minimum radius of the neck is normalized by the location \(z^*\), where the pinch-off takes
place. The minimum radius of the neck decreases with time and the exponent of time shifts from $2/3$ to $1$. The exponents of $2/3$ and $1$ correspond to the potential flow scaling law [25] and the viscous scaling law [19], respectively. Since the length scale becomes very small close to the pinch-off, the viscous effect overcomes the inertial effect and the scaling alters from the potential flow scaling law to the viscous scaling law. Hence we expect the viscous scaling law can apply the local dynamics of the thread pinch-off.

### 4.2 Motion of Thin Thread

The motion of the thin thread formed after pinch-off is similar to the motion of the ligament. While the resolution is not enough to capture the precise motion of the thin thread, the thread breaks up into several satellite drops. Thus the motion of the thread can be discussed equivalently if the scale is similar to that for the discussion in Sec. 3 \[O(10^{-3}) < Oh_d < O(10^{-1})\].

**Table 1** shows the Ohnesorge number for the thread formed from the ligament of $Oh_d=5.9 \times 10^{-2}$, $\mu_c/\mu_d=1$. The radius of the thread is $3$ to $6$ % of the initial radius of the ligament. The order of magnitude of the Ohnesorge number is $10^{-1}$. Hence the order is within the region of our study and the same discussion is applied to the thread motion and the appropriate range of the Ohnesorge number is $O(10^{-1}) < Oh_d < O(10^{-2})$ for the ligament.

### 6. ACKNOWLEDGMENT

We are grateful to Professor G. Tryggvason for providing a front-tracking code and discussion about the results.

### 7. NOMENCLATURE

- **A** area
- **a** radius of the ligament
- **l** length of the ligament
- **n** unit normal
- **Oh** Ohnesorge number, $\mu/\sqrt{\rho \sigma}$
- **P** pressure
- **r** radial coordinate
- **t** time
- **u** velocity vector
- $\bar{v}$ time averaged velocity
- **x** position vector
- **z** axial coordinate

**Greek letters**

- $\delta$ delta function, amplitude of disturbance
- $\kappa$ curvature
- $\mu$ viscosity
- $\rho$ density
- $\sigma$ interfacial tension
- $\omega$ growth rate of the most unstable disturbance

**Subscripts and superscript**

- $0$ initial $(t=0)$
- $c$ external fluid (continuous phase)
- $d$ drop or ligament (dispersed phase)
- $f$ interface (front)
- **min** minimum point
- $r$ retraction
- $T$ transpose
- * pinch-off point
- $\mu$ viscous
- $\rho$ inertial

into daughter drops or relaxes back to a spherical drop. These two modes are determined by the Ohnesorge number, the viscosity ratio, and the initial length of the ligament.

The necking process, which affects the formation of satellite drops, is examined. The neck is thinning down with the viscous scaling law close to the pinch-off point and the pinch-off of the thread can be predicted by the law.

The motion of the thread is discussed. If the thread breaks up into satellite drops with similar manner to the parental ligament, the same discussion is applied to the thread motion and the appropriate range of the Ohnesorge number is $O(10^{-1}) < Oh_d < O(10^{-2})$ for the ligament.
REFERENCES


